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2008 J. Phys. A: Math. Theor. 41 215004

(http://iopscience.iop.org/1751-8121/41/21/215004)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 215004 (11pp) doi:10.1088/1751-8113/41/21/215004

Continuous time random walks in finite domains and general boundary conditions: some formal considerations

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Received 1 February 2008, in final form 13 April 2008
Published 9 May 2008
Online at stacks.iop.org/JPhysA/41/215004

Abstract

The present work studies continuous time random walks (CTRWs) in a finite domain. A broad class of boundary conditions, of which absorbing and reflecting boundaries are particular cases, is considered. It is shown how any CTRW in this class can be mapped to a CTRW in an infinite domain. This may allow applying well-known techniques for infinite CTRWs to the problem of obtaining the fluid limit for finite domain CTRWs, where the *fluid limit* (or hydrodynamic limit) refers to the partial differential equation describing the long time and large distance behaviour of the system. As an illustration, the fluid limit equation and its propagator are obtained explicitly in the case of purely reflecting boundaries. We also derive the modification of the Riemann–Liouville fractional differential operators implementing the reflecting boundary conditions.

PACS numbers: 05.40.Fb, 02.50.Ey, 05.60.Cd, 05.10.Gg

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Scale-free transport arises in many physical, biological and social systems [1–4]. This phenomenon occurs when the underlying microscopic transport processes lack a characteristic scale, leading to transport equations at the macroscopic level that differ from the classical diffusion equations. The issue is best understood in the framework of continuous time random walks (CTRWs) [5, 6], which are models describing the effective mesoscopic kinetics (of particles or other conserved entities) probabilistically. The broad scope of mesoscopic kinetics accessible by the CTRW description provides a method to explore transport processes beyond the classical paradigm of diffusion.

The simplest realization of a CTRW is the separable one. It is defined in terms of two probability distribution functions (pdfs): a *step size pdf* $p(x, x')$, giving the probability of making a jump from x' to x, and a *waiting time pdf* $\psi(t - t')$, giving the probability of not moving for a time $t - t'$ between two consecutive jumps. When the step size pdf is Gaussian and the waiting time pdf is exponential, the fluid (long time, large distance) limit yields diffusive equations [3, 4]. These choices intrinsically imply the existence of finite characteristic length and time scales, associated with the underlying microscopic transport mechanism. CTRW models lacking characteristic scales can be constructed by choosing p and/or ψ from the family of stable Lévy distributions [7]. In this case, the fluid or hydrodynamic limit is expressed in terms of transport equations that contain fractional differential operators (generalizations of integer derivatives that are essentially non-local) in space, time or both [8]. Whereas CTRWs and their connection with fractional differential equations have been studied extensively in infinite domains [3, 9–11], their finite domain counterparts have mostly been studied only as lattice models [5, 12, 13].

The interest in investigating finite-size effects is obvious from the point of view of practical applications, which always describe transport processes in a finite domain. When p is a Gaussian (and narrow) distribution, we know that the influence of the system boundaries on the shape of the solution is negligible at positions sufficiently far from the boundaries (a few widths of the Gaussian). However, when p is a Lévy distribution with algebraic tails, the influence of the system boundaries is 'felt' throughout the system [14], and a proper treatment of the boundary is therefore essential for advancing the understanding of transport in finite systems.

Finite domain CTRWs with periodic boundary conditions were studied in [15], where the corresponding fractional derivative operators on the circle were derived. The key to the success of that study was the exploitation of the correspondence of these periodic CTRWs with an infinite domain CTRW. Well-understood mathematical techniques are available for the latter case [16], allowing the derivation of the fluid limit equations and the propagator of the periodic problem. These fluid limit equations provide a description of the system behaviour at long times and distances, in the same way that the classical diffusive equation does so in the case of the standard Gaussian and exponential CTRW mentioned above. In the same spirit, the present work deals with CTRWs on the interval $[-L, L]$ with a broad class of boundary conditions that interpolate between the commonly used reflecting and absorbing boundary conditions [17]. We show that CTRWs of this class can formally be mapped to an infinite domain CTRW (which is not necessarily probability conserving) under specific but ample conditions. As with the periodic CTRW, this reformulation may allow the application of techniques developed for infinite domain CTRWs to derive fluid limit equations for finite systems. As an illustration, we explicitly work out the fluid limit equation in the special case of two perfectly reflecting boundaries, and compute the modified form of the fractional derivative operators, extending the results of [18].

The rest of the paper is organized as follows: in section 2, we develop the general formalism for handling CTRWs on a finite domain with arbitrary boundary conditions. In section 3, we specialize to purely reflecting boundaries, show how this case relates to the periodic CTRW, obtain some analytic results and derive the corresponding fractional differential operator. Finally, in section 4 we draw some conclusions.

2. CTRW on a finite interval with general boundary conditions

We start by considering a one-dimensional, separable, translationally invariant, Markovian CTRW on the real line. Its dynamics are described in terms of a generalized master equation

(GME) of the form [19]

$$\partial_t n(x,t) = \int_{-\infty}^{\infty} p(x-x')n(x',t)\,\mathrm{d}x' - n(x,t), \qquad x \in [-\infty, \infty]. \tag{1}$$

Here, $n(x,t)$ is the particle density, p is the step size pdf and t is the time, which has been normalized to the mean waiting time. The natural way to restrict the CTRW to a finite interval $[-L, L]$ is by truncating the integral:

$$\partial_t n(x,t) = \int_{-L}^{L} \bar{p}(x,x')n(x',t)\,\mathrm{d}x' - n(x,t), \qquad x \in [-L, L]. \tag{2}$$

However, if we maintain the same step size pdf as in the infinite CTRW (i.e., set $\bar{p} = p$), this implies that equation (2) has been endowed with fully absorbing boundary conditions at $\pm L$: any particle jumping out of this interval is lost. And it also implies that the restricted CTRW and its associated GME are no longer probability conserving. Even so, we will keep referring to them as a CTRW and a GME in the following. In order to implement other types of boundary conditions, the restriction of the integral to the interval $[-L, L]$ must be supplemented with a proper modification of the step size pdf consistent with the desired boundary conditions. In what follows, this modified step size pdf will be denoted as $\bar{p}$.

The class of boundary conditions considered in this work is parameterized by $0 \leqslant r_L, r_R \leqslant 1$, which act as reflection coefficients at the left and right boundaries, respectively. More precisely, these coefficients specify the probability with which a given particle is reflected at the boundary (whereas the probability of being absorbed is given by $1 - r_L$ and $1 - r_R$). Thus, if both r_L and r_R are non-zero, any particle has a non-zero probability to be reflected an infinite number of times between the two boundaries. In other words, for the particle to go from x' to x, there are an infinite number of paths. In order to enumerate these paths, we subdivide them into paths that leave x' to the left (*L*-type) and to the right (*R*-type). When the total number of reflections is $m \geqslant 1$, then the total path Δx is

$$\begin{aligned} (\Delta x)_{L\text{-type}} &= (-1)^m x - x' - 2mL, \\ (\Delta x)_{R\text{-type}} &= (-1)^m x - x' + 2mL. \end{aligned} \tag{3}$$

This device allows us to calculate the step size pdf $\bar{p}$ resulting from the imposition of such boundary conditions at $\pm L$ on the infinite domain CTRW, characterized by $p(x, x')$. It is constructed by summing over all possible paths while 'attenuating' each reflection by the appropriate reflection constant (i.e., multiplying by the probability of being reflected). This total probability is (with $x, x' \in [-L, L]$)

$$\begin{aligned} \overline{p}(x,x') = p(x-x') \\ + \sum_{m=1}^{\infty} \left[r_L^{m_+} r_R^{m_-} p((-1)^m x - x' - 2mL) + r_L^{m_-} r_R^{m_+} p((-1)^m x - x' + 2mL) \right], \end{aligned} \tag{4}$$

where $m_- = \lfloor m/2 \rfloor, m_+ = \lfloor (m+1)/2 \rfloor$ and $\lfloor \cdot \rfloor$ stands for the integer part. This equation describes all possible cases and combinations of absorbing, reflecting and mixed boundary conditions. When $r_L = r_R$, the expression simplifies considerably due to the fact that $m_- + m_+ = m$.

We can condensate this expression into a single sum by associating *L*-type reflections with negative m and *R*-type reflections with positive m (and the direct route with $m = 0$):

$$\overline{p}(x,x') = \sum_{m=-\infty}^{\infty} r_L^{m_L} r_R^{m_R} p((-1)^m x - x' + 2mL), \tag{5}$$

where $m_L = \lfloor(|m| + H(-m))/2\rfloor$, $m_R = \lfloor(|m| + H(m))/2\rfloor$ and H is the Heaviside function ($H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geqslant 0$).

We benefit from separating even and odd terms:

$$\overline{p}(x, x') = \sum_{m=-\infty}^{\infty} r_L^{|m|} r_R^{|m|} p(x - x' - 4mL) + \sum_{m=-\infty}^{\infty} r_L^{|m|} r_R^{|m+1|} p(-x - x' + 2L + 4mL). \tag{6}$$

Inserting (6) in equation (2):

$$\partial_t n(x, t) = \sum_{m=-\infty}^{\infty} \int_{-L}^{L} r_L^{|m|} r_R^{|m|} p(x - x' - 4mL, x') n(x', t)\, \mathrm{d}x' + \sum_{m=-\infty}^{\infty} \int_{-L}^{L} r_L^{|m|} r_R^{|m+1|} p(-x - x' + 2L + 4mL, x') n(x', t)\, \mathrm{d}x' - n(x, t). \tag{7}$$

As pointed out above, the GME (2) is defined only within $[-L, L]$. Performing the change of variables $u = x' + 4mL$ in the first term of the right-hand side, and $u = -x' + 2L + 4mL$ in the second one, we obtain:

$$\partial_t n(x, t) = \sum_{m=-\infty}^{\infty} \int_{-L+4mL}^{L+4mL} r_L^{|m|} r_R^{|m|} p(x - u) n(u - 4mL, t)\, \mathrm{d}u + \sum_{m=-\infty}^{\infty} \int_{L+4mL}^{3L+4mL} r_L^{|m|} r_R^{|m+1|} p(-x + u) n(-u + 2L + 4mL, t)\, \mathrm{d}u - n(x, t), \tag{8}$$

where u (but not x!) now spans the whole real line. This expression suggests that it may be possible to extend the definition of n to the whole real line in a consistent manner.

In this framework, the following definitions are useful:

$$\begin{aligned} m(x) &= \lfloor(x + L)/4L\rfloor, \\ s(x) &= \mathrm{sign}(x - L - 4m(x)L). \end{aligned} \tag{9}$$

The parameter m enumerates intervals of x with length $4L$, while s indicates whether x lies in the first or second half of such an interval. Equation (8) suggests that the extension of $n(x, t)$ should be

$$N(x, t) = n(-s(x)(x - 4m(x)L) + (s(x) + 1)L, t). \tag{10}$$

Thus, for $x \in [-L, L]$ we have $N(x, t) = n(x, t)$, and $n(x, t)$ is extended outside $[-L, L]$ so that $N(x, t)$ satisfies

$$\begin{aligned} N(x + 4L, t) &= N(x, t), \\ N(x + 2L, t) &= N(-x, t). \end{aligned} \tag{11}$$

The corresponding extension of p is

$$P(x, x') = r_L^{|m(x') - m(x)|} r_R^{|m(x') - m(x) + (s(x') - s(x))/2|} p(-s(x')(x - x')). \tag{12}$$

We note that when $p(x - x') = p(x' - x)$, then $P(x, x') = P(x', x)$. An example of the shape of P is shown in figure 1.

The GME can now be written as

$$\partial_t N(x, t) = \int_{-\infty}^{\infty} P(x, x') N(x', t)\, \mathrm{d}x' - N(x, t), \qquad x \in \mathbb{R}. \tag{13}$$

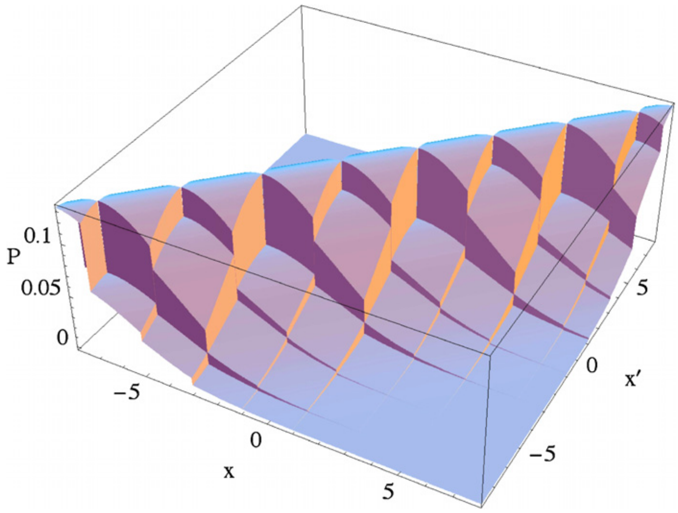

Figure 1. $P(x, x')$ for a Gaussian distribution p (width $\sigma = 0.1$, $L = 1$, $r_L = 0.8$, $r_R = 0.5$).

To check that this extension agrees with equation (8) for $x \in [-L, L]$ (i.e. $m(x) = 0, s(x) = -1$) we write

$$P(x,x') = \begin{cases} r_L^{|m(x')|} r_R^{|m(x')|} p(x - x') & \text{if} \quad x' \in [(4m(x') - 1)L, \quad (4m(x') + 1)L), \\ r_L^{|m(x')|} r_R^{|m(x')+1|} p(x' - x) & \text{if} \quad x' \in [(4m(x') + 1)L, \quad (4m(x') + 3)L). \end{cases} \tag{14}$$

Using

$$N(x,t) = \begin{cases} n(x - 4m(x)L, t) & \text{if} \quad x \in [(4m(x) - 1)L, \quad (4m(x) + 1)L), \\ n(-x + 2L + 4m(x)L, t) & \text{if} \quad x \in [(4m(x) + 1)L, \quad (4m(x) + 3)L), \end{cases} \tag{15}$$

and inserting equation (14) in equation (13), it is not difficult to see that equation (8) is recovered.

The non-trivial point is whether equation (13) is self-consistent. We must prove that the properties (11) are conserved for a given initial distribution $N(x, t)$ satisfying these same properties. Noting that $u \mapsto u + 4L$ implies $s(u) \mapsto s(u)$ and $m(u) \mapsto m(u) + 1$, it is easy to show that

$$P(x + 4L, x') = P(x, x' - 4L). \tag{16}$$

Analogously, $u \mapsto u + 2L$ implies $s(u) \mapsto -s(u)$ and $m(u) \mapsto m(u) + (1 + s(u))/2$, while $u \mapsto -u$ implies $s(u) \mapsto s(u)$ and $m(u) \mapsto -m(u) - (1 + s(u))/2$. Then,

$$P(x + 2L, x') = P(-x, -x' + 2L). \tag{17}$$

First we show that $\partial_t N(x + 4L, t) = \partial_t N(x, t)$:

$$\begin{aligned} \partial_t N(x + 4L, t) &= \int_{-\infty}^{\infty} P(x + 4L, x') N(x', t)\, \mathrm{d}x' - N(x + 4L, t) \\ &= \int_{-\infty}^{\infty} P(x, x' - 4L) N(x', t)\, \mathrm{d}x' - N(x + 4L, t) \\ &= \int_{-\infty}^{\infty} P(x, x') N(x' + 4L, t)\, \mathrm{d}x' - N(x + 4L, t) \end{aligned}$$

$$= \int_{-\infty}^{\infty} P(x, x')N(x', t)\,\mathrm{d}x' - N(x, t)$$
$$= \partial_t N(x, t). \tag{18}$$

Next, we show that $\partial_t N(x + 2L, t) = \partial_t N(-x, t)$:

$$\begin{aligned} \partial_t N(x + 2L, t) &= \int_{-\infty}^{\infty} P(x + 2L, x')N(x', t)\,\mathrm{d}x' - N(x + 2L, t) \\ &= \int_{-\infty}^{\infty} P(-x, -x' + 2L)N(x', t)\,\mathrm{d}x' - N(x + 2L, t) \\ &= \int_{-\infty}^{\infty} P(-x, x')N(-x' + 2L, t)\,\mathrm{d}x' - N(x + 2L, t) \\ &= \int_{-\infty}^{\infty} P(-x, x')N(x', t)\,\mathrm{d}x' - N(-x, t) \\ &= \partial_t N(-x, t). \end{aligned} \tag{19}$$

In summary, we have provided an extension of the finite domain GME with arbitrary boundary conditions to a GME on an infinite domain. The boundary conditions of the finite domain GME translate into a modification of the kernel of the corresponding infinite domain GME. The purely reflecting case is special in the sense that the corresponding form of P is quite simple (cf equation (12)), and that it is probability conserving. This case will be studied in the following section, since it can be solved analytically in a few cases.

3. Purely reflecting boundaries

In the case $r_L = r_R = 1$ one obtains $P(x, x') = p(-s(-x')(x - x'))$, according to equation (12). Since p is a true probability distribution (it is positive and its integral is 1), so is P. If, in addition, p is symmetric (i.e., $p(u) = p(-u)$), we have $P(x, x') = p(x - x')$. Hence, the extended GME (13) simply reads

$$\partial_t N(x, t) = \int_{-\infty}^{\infty} p(x - x')N(x', t)\,\mathrm{d}x' - N(x, t). \tag{20}$$

The information that equation (20) actually comes from a CTRW formulated on $[-L, L]$ enters through the initial condition, which must satisfy

$$N(x + 4L, 0) = N(x, 0), \qquad N(x + 2L, 0) = N(-x, 0). \tag{21}$$

The time evolution preserves these properties, as proven in section 2.

Let us work out the fluid limit propagator of (20). Since $N(x, t)$ is periodic in x with period $4L$ we can proceed along the same lines as in [15]. Let $f(x)$ be periodic with period $4L$. We use the following convention for its expansion in Fourier series:

$$f(x) = \frac{1}{4L} \sum_{m=-\infty}^{\infty} f_m\,\mathrm{e}^{-\mathrm{i}\pi m x/2L}. \tag{22}$$

Let $\Lambda(k)$ be the characteristic exponent of p, i.e. $\hat{p}(k) = \exp \Lambda(k)$. The fluid limit corresponds to the limit of small k, and is obtained by approximating $\hat{p}(k) \approx 1 + \Lambda(k)$. As shown in [15] the fluid limit of (20) in Fourier space is

$$\partial_t N_m(t) = \Lambda(m)N_m(t), \tag{23}$$

with the solution

$$N_m(t) = N_m(0)\,\mathrm{e}^{\Lambda(m)t}. \tag{24}$$

Thus,

$$N(x,t) = \frac{1}{4L}\sum_{m=-\infty}^{\infty} N_m(0)\,\mathrm{e}^{\Lambda(m)t}\,\mathrm{e}^{-\mathrm{i}\pi mx/2L}. \tag{25}$$

We will consider the case in which $\Lambda(k)$ is the characteristic exponent of a symmetric Lévy distribution:

$$\Lambda(k) = -\sigma^\alpha |k|^\alpha, \qquad \alpha \in (0,2]. \tag{26}$$

Take an initial condition $n(x,0) = \delta(x-x_0)$, $x_0 \in [-L, L]$. Its extension, $N(x,0)$, is

$$N(x,0) = \sum_{m=-\infty}^{\infty} \delta(x - x_0 + 4mL) + \sum_{m=-\infty}^{\infty} \delta(x + x_0 - 2L - 4mL), \tag{27}$$

with Fourier expansion,

$$N(x,0) = \frac{1}{4L}\sum_{m=-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\pi mx_0/2L} + \mathrm{e}^{\mathrm{i}\pi m(-x_0+2L)/2L})\,\mathrm{e}^{-\mathrm{i}\pi mx/2L}. \tag{28}$$

Hence,

$$N_m(0) = \mathrm{e}^{\mathrm{i}\pi mx_0/2L} + \mathrm{e}^{\mathrm{i}\pi m(-x_0+2L)/2L}. \tag{29}$$

And from (25) we find the expression of the propagator:

$$N(x,t) = \frac{1}{4L}\sum_{m=-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}\pi mx_0/2L} + \mathrm{e}^{\mathrm{i}\pi m(-x_0+2L)/2L})\,\mathrm{e}^{-\sigma^\alpha|k|^\alpha t}\,\mathrm{e}^{-\mathrm{i}\pi mx/2L}. \tag{30}$$

Beautifully, the propagator of the fluid equation can be seen as the superposition (or interference) of two propagators of the CTRW on a circle of length $4L$. The first (resp. second) term on the RHS of (30) corresponds to a periodic propagator initially located at x_0 (resp. $-x_0+2L$). In particular, for a symmetric Cauchy distribution, $\hat{p}(k) = \exp(-\sigma|k|)$, we can give an explicit solution by using the results of [15]:

$$N(x,t) = \frac{\sinh(t/\rho)}{4L}\left[\frac{1}{\cosh(t/\rho) - \cos\left(\frac{\pi(x-x_0)}{2L}\right)} + \frac{1}{\cosh(t/\rho) + \cos\left(\frac{\pi(x+x_0)}{2L}\right)}\right], \tag{31}$$

where $\rho = \frac{2}{\pi}\frac{L}{\sigma}$. Observe that the translational invariance of the propagator is broken by the non-trivial dependence on x_0. Of course, $N(x,t) \to (2L)^{-1}$ when $t \to \infty$, as required by the conservation of the number of particles.

When $x_0 = (2m+1)L$, the two terms on the RHS of (31) are in phase and the propagator is

$$N(x,t) = \frac{1}{2L}\frac{\sinh(t/\rho)}{\cosh(t/\rho) + (-1)^{m+1}\sin\left(\frac{\pi x}{2L}\right)}. \tag{32}$$

If $x_0 = 2mL$ the two terms on the RHS of (31) are in antiphase and the propagator has periodicity $2L$:

$$N(x,t) = \frac{1}{2L}\frac{\sinh(2t/\rho)}{\cosh(2t/\rho) - \cos(\pi x/L)}. \tag{33}$$

Some graphs illustrating the time evolution of $N(x,t)$ are shown in figures 2–4.

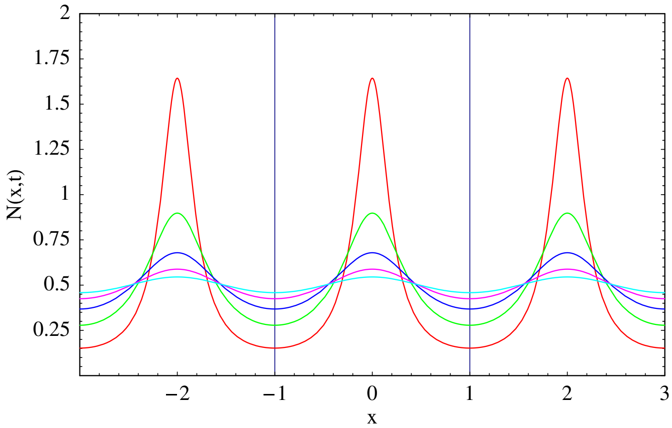

Figure 2. $N(x,t)$ for a Cauchy distribution p ($\sigma = 0.1, L = 1, x_0 = 0$), for times $t = 2, 4, 6, 8, 10$.

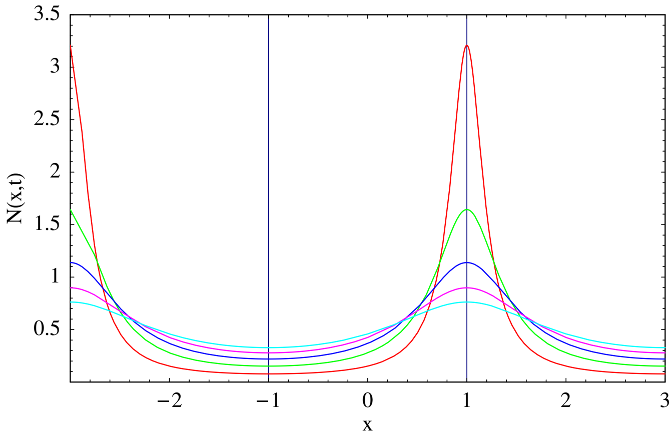

Figure 3. $N(x,t)$ for a Cauchy distribution p ($\sigma = 0.1, L = 1, x_0 = 1$), for times $t = 2, 4, 6, 8, 10$.

3.1. Fractional differential operators with reflecting boundary conditions

It is interesting to give the explicit form of equation (23) in x-space and for $n(x,t)$ when $\Lambda(k)$ corresponds to a Lévy distribution. Recalling that for reflecting boundaries the evolution of $N(x,t)$ is governed by (20), it is a well-known result ([3, 4]) that the fluid limit equation when p is a symmetric Lévy distribution is given by

$$\partial_t N(x,t) = -\frac{\sigma^\alpha}{2\cos(\pi\alpha/2)}\left({}_{-\infty}D^\alpha + {}^{\infty}D^\alpha\right) N(x,t), \tag{34}$$

where ${}_{-\infty}D^\alpha$ and ${}^{\infty}D^\alpha$ are the Riemann–Liouville operators defined in (B.1).

Using properties (11) and after some manipulations analogous to those of [15] one finds the fractional differential equation satisfied by $n(x,t)$. Namely,

$$\partial_t n(x,t) = -\frac{\sigma^\alpha}{2\cos(\pi\alpha/2)}\left({}_{-L}\mathcal{D}^\alpha + {}^{L}\mathcal{D}^\alpha\right) n(x,t). \tag{35}$$

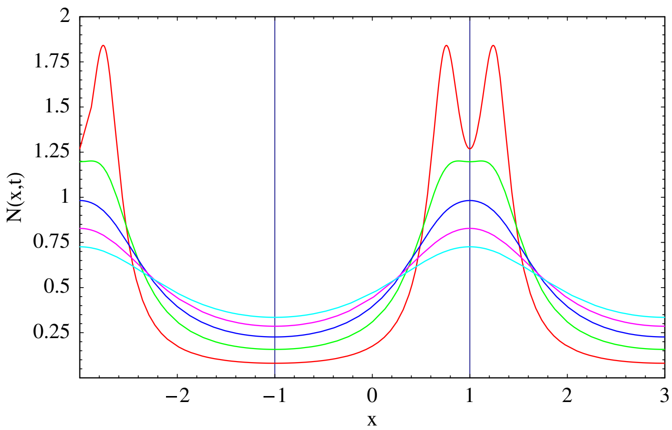

Figure 4. $N(x, t)$ for a Cauchy distribution p ($\sigma = 0.1$, $L = 1$, $x_0 = 0.75$), for times $t = 2, 4, 6, 8, 10$.

The modified Riemann–Liouville operators are

$$
\begin{aligned}
{}_{-L}\mathcal{D}^{\alpha} f(x) &= {}_{-L}D^{\alpha} f(x) \\
&\quad + \lambda \int_{-L}^{L} f(x') \left[\zeta\left(1+\alpha, 1 + \frac{x - x'}{4L}\right) + \zeta\left(1+\alpha, \frac{1}{2} + \frac{x + x'}{4L}\right)\right] \mathrm{d}x', \\
{}^{L}\mathcal{D}^{\alpha} f(x) &= {}^{L}D^{\alpha} f(x) \\
&\quad - \lambda \int_{-L}^{L} f(x') \left[\zeta\left(1+\alpha, 1 + \frac{x' - x}{4L}\right) + \zeta\left(1+\alpha, \frac{3}{2} - \frac{x + x'}{4L}\right)\right] \mathrm{d}x', \qquad (36)
\end{aligned}
$$

where $\lambda = 1/(\Gamma(-\alpha)(4L)^{\alpha+1})$, and ζ is the Hurwitz zeta function,

$$\zeta(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}, \qquad \mathrm{Re}(s) > 1, \qquad m + a \neq 0. \qquad (37)$$

Equation (36) explicitly shows the modification of the fractional differential operators due to the presence of reflecting boundary conditions.

4. Conclusions

We have shown how the CTRW on a finite domain with general boundary conditions (absorbing, reflecting or mixed) can be mapped to a (not necessarily probability conserving) CTRW on the infinite domain. The boundary conditions corresponding to the finite domain CTRW are absorbed into the kernel of the GME of the infinite domain CTRW. This transformation may open up the possibility of applying techniques developed for infinite CTRWs to the calculation of fluid limit equations, propagators and other information for CTRWs restricted to a finite domain.

As an application of the formalism, we have worked out the fluid limit of the case with perfectly reflecting boundary conditions. In the process, we have derived an explicit form of the fractional differential equation for the reflecting boundary case, illustrating that the presence of a boundary in a finite system leads to the modification of the fractional differential operators involved. Finally, we have computed the propagator of the fluid limit equation. It turns out that the solutions of the CTRW with reflecting boundaries on an interval of length $2L$ correspond to symmetrized sums of the solutions of the CTRW on a circle of length $4L$.

Acknowledgments

This research was sponsored in part by DGICYT (Dirección General de Investigaciones Científicas y Tecnológicas) of Spain under project no. ENE2006-15244. Part of this research was sponsored by the Laboratory Research and Development Program of the Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the US-DOE under contract number DE-AC05-00OR22725. The figures in this paper were prepared using the *Mathematica* software package [20].

Appendix A. Lévy skew alpha-stable distributions

The family of *Lévy skew alpha-stable distributions* (or simply *stable distributions*, or *Lévy distributions*) is parameterized by four real numbers $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\sigma > 0$ and $\mu \in \mathbb{R}$. Their characteristic function (i.e. their Fourier transform) is given by [7]:

$$\hat{S}(\alpha,\beta,\sigma,\mu)(k) = \begin{cases} \exp\left(-\sigma^\alpha |k|^\alpha \left[1 - \mathrm{i}\beta \,\mathrm{sign}(k) \tan\left(\frac{\pi\alpha}{2}\right)\right] + \mathrm{i}\mu k\right) & \alpha \neq 1, \\ \exp\left(-\sigma|k| \left[1 + \mathrm{i}\beta \frac{2}{\pi} \mathrm{sign}(k) \ln|k|\right] + \mathrm{i}\mu k\right) & \alpha = 1. \end{cases} \quad \text{(A.1)}$$

According to the generalized central limit theorem ([7, 21]), stable distributions are the only possible distributions with a domain of attraction. The *index of stability*, α, is related to the asymptotic behaviour of $S(\alpha,\beta,\sigma,\mu)(x)$ at large x:

$$S(\alpha,\beta,\sigma,\mu)(x) = \begin{cases} C_\alpha \left(\frac{1-\beta}{2}\right) \sigma^\alpha |x|^{-1-\alpha} & x \to -\infty, \\ C_\alpha \left(\frac{1+\beta}{2}\right) \sigma^\alpha |x|^{-1-\alpha} & x \to \infty, \end{cases} \quad \text{(A.2)}$$

for $\alpha \in (0,2)$. For $\alpha = 2$, $S(2,\beta,\sigma,\mu)$ is a Gaussian distribution.

The *skewness parameter*, β, measures the asymmetry of the distribution. If $\beta = 0$, $S(\alpha,0,\sigma,\mu)$ is symmetric with respect to μ.

Appendix B. Fractional differential operators on the real line

The *Riemann–Liouville fractional differential operators* are defined as [22]:

$$\begin{aligned} {}_aD^\alpha f(x) &:= \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_a^x \frac{f(x')}{(x-x')^{\alpha-m+1}} \,\mathrm{d}x', \\ {}^bD^\alpha f(x) &:= \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_x^b \frac{f(x')}{(x'-x)^{\alpha-m+1}} \,\mathrm{d}x', \end{aligned} \quad \text{(B.1)}$$

where Γ is the Euler Gamma function and m is the integer verifying $m-1 \leqslant \alpha < m$. When $a \to -\infty$ and $b \to \infty$, the action of the Riemman–Liouville operators in Fourier space satisfies the important property:

$$\begin{aligned} &\mathcal{F}\left[{}_{-\infty}D^\alpha f\right](k) = (-\mathrm{i}k)^\alpha \mathcal{F}[f](k) \\ &\mathcal{F}[{}^\infty D^\alpha f](k) = (\mathrm{i}k)^\alpha \mathcal{F}[f](k), \end{aligned}$$

where $\mathcal{F}[\cdot]$ stands for the Fourier transform. This property justifies the name *fractional differential operators* because they generalize the notion of differentiation to non-integer order.

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